UW/PT-94-10

MOTION OF SPIN-1/2 PARTICLES IN EXTERNAL GRAVITATIONAL AND ELECTROMAGNETIC FIELDS

Stamatis Vokos*
Physics Department, FM-15
University of Washington
Seattle, WA 98195
USA

ABSTRACT

We study the semi-classical limit of the solution of the Dirac equation in a background electromagnetic/gravitational plane wave. We show that the exact solution corresponding to an asymptotically fixed incoming momentum satisfies constraints consistent with the classical notion of a spinning particle. In order to further analyze the motion of a spinning particle in this external inhomogeneous field one has to consider wave-packet superpositions of these exact solutions. We are currently investigating the existence of a *classical* theory of a phenomenological spin tensor which reproduces our quantum-mechanical results.

To appear in the proceedings of the
7th Marcel Grossmann Meeting
on General Relativity (MG 7)
Stanford, CA, July 24 – 30, 1994

*E-Mail: vokos@phys.washington.edu

1. Introduction

There has been a resurgence of interest lately in the study of the dynamics of a spinning object in an external field. Indeed, there are special cases where the answer is well-known, examples of which are the case of a microscopic particle in a (nearly) uniform electromagnetic field and the formal description of a macroscopic spin in a gravitational field with special symmetries. The former case is described by the Michel-Bargmann-Telegdi (BMT) equation, while the latter is formally solved within the Papapetrou-Dixon theoretical framework. One may argue that the general case is well-defined, since the coupling of spinning matter with external fields is known to be through the appropriate field equation in the given background field. This approach, however, is lacking in practicality. The exact solutions of any field equation in a background field are very few and most do not correspond to physically realistic configurations. Clearly, one does not resort to solving the Dirac equation in order to predict the behavior of an electron spin in an accelerator (cf. Siberian snakes)! One is then drawn to the challenging task of describing this interaction of spinning particles in terms of a set of self-consistent equations for the components of a phenomenological spin tensor which satisfy certain constraints. The approaches to this problem have been varied (see Ref. 1 and references therein). Researchers have employed arguments of analogy with the BMT equation, have used Lagrangian theories, Hamiltonian constructions, Ruthian formulations, supersymmetric manipulations, quaternion and octonion algebras, etc.

In this talk, I motivate the use of a specific toy model where one may be able to test the extant theories of spin-$\frac{1}{2}$ particles in external fields. My philosophical bias is that *no* (semi-)classical theory of spinning particles in external fields can ever hope to be acceptable if it does not agree with a (properly chosen) classical limit of the corresponding field equation. It is useful, therefore, to start with the Dirac equation and construct a suitable phenomenological spin tensor, whose time development is governed by equations deduced from the quantum mechanical dynamics.

2. The model

There exists a particularly simple (but non-trivial), realistic gravitational and electromagnetic background, where one can solve the Dirac equation exactly, namely in a plane gravitational or electromagnetic wave. The wave's spectral decomposition can be arbitrary, guaranteeing departure from uniformity. The high degree of symmetry, however, provides one with a wonderfully tractable solution. The solution in the electromagnetic case is originally due to Volkov. Here we lean heavily on the description of Brown.[2] Due to the formal similarity of the electromagnetic and gravitational backgrounds, we were able to easily obtain a solution for the gravitational case, as well. In the interest of brevity, we present here the outline of the argument in the electromagnetic case only.

A plane wave is given by a vector potential A_μ which depends on a single argument only, i.e.

$$A_\mu = A_\mu(y)\,, \tag{1}$$

where $y = n_\mu x^\mu$, and n^μ is a null vector which describes the unique direction of propagation of the plane wave. In the Lorentz gauge $\partial \cdot A = 0 = n \cdot A$, one can obtain the solution of the Dirac equation

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi_p(x) = 0\,, \tag{2}$$

where the subscript p in the solution refers to the asymptotic incoming momentum of the fermion. The explicit form of the solution[b]together with the accompanying details are given elsewhere.[1,2]

One can perform a Gordon decomposition of the Dirac current, $j_p^\mu(x)$,

$$\begin{aligned} j_p^\mu(x) &= \overline{\psi_p}(x)\gamma^\mu\psi_p(x) \equiv j_{\mathbf{c}}{}^\mu(x) + j_{\mathbf{s}}{}^\mu(x) \\ &= \frac{i\hbar}{2m}\left(\overline{\psi_p}(x)D^\mu\psi_p(x) - D^\mu\overline{\psi_p}(x)\psi_p(x)\right) + \frac{\hbar}{2m}\partial_\nu\left(\overline{\psi_p}(x)\sigma^{\mu\nu}\psi_p(x)\right)\,. \end{aligned} \tag{3}$$

The first term in the above decomposition is the convective current and the second is the spin current. Both currents are separately conserved. One obtains,

$$j_{\mathbf{c}}{}^\mu(y) = \frac{1}{m}\left(p^\mu - eA^\mu(y) + n^\mu I_p(y)\right) + \frac{e\hbar}{4m}\frac{n^\mu}{n\cdot p}\overline{u}(p)F_{\alpha\beta}(y)\sigma^{\alpha\beta}u(p) \tag{4}$$

$$j_{\mathbf{s}}{}^\mu(y) = -\frac{e\hbar}{4m}\frac{n^\mu}{n\cdot p}\overline{u}(p)F_{\alpha\beta}\sigma^{\alpha\beta}u(p) \equiv \frac{1}{m}\partial_\nu J^{\mu\nu}\,, \tag{5}$$

where $I_q(y) = \frac{1}{2n\cdot q}(2eA\cdot q - e^2A^2)$ and $u(p)$ is a free Dirac spinor. Note that each current depends only on y, and that the total current is independent of the spin, through a non-trivial cancellation of terms.[c]

A natural candidate for a semi-classical spin tensor is $\mathcal{S}^{\mu\nu}$, where

$$J^{\mu\nu}(y) = \overline{\psi_p}(x)\frac{\hbar}{2}\sigma^{\mu\nu}\psi_p(x) \equiv \overline{u}(p)\mathcal{S}^{\mu\nu}(y)u(p)\,. \tag{6}$$

Exploiting the classical equality $y = \frac{n\cdot p}{m}\tau$, we are then able to express the spin tensor of the fermion at proper time τ in terms of the initial spin tensor at $\tau \to -\infty$,

$$\mathcal{S}^{\mu\nu}(\tau) = \mathcal{O}^{\mu\nu}{}_{\rho\lambda}(\tau)\mathcal{S}^{\rho\lambda}(-\infty)\,, \tag{7}$$

[b] One interesting point to note here, however, is that one can arrive at the solution in the following way. One solves the classical Lorentz force equation (or geodesic equation, in the gravitational case), in terms of Hamilton-Jacobi theory, and one obtains explicitly Hamilton's characteristic function, ϕ. Armed with ϕ, one can show that the exponential of ϕ, $\exp(i\phi/\hbar)$, satisfies the Klein-Gordon equation in the relevant background field, i.e. the WKB approximation is exact in this case. The final step is to verify that the solution to the Dirac equation in the same background is equal to the WKB factor multiplied by an appropriate spinor. The beauty of this approach, is that one is always in touch with the classical interpretation of quantum mechanical observables. One example is the establishment of the proportionality between $y = n\cdot x$ and the classical proper time τ, $y \propto \tau$.

[c] This shows the error in the usual statement that the convective current is the part of the current which corresponds to the translational motion only and is, therefore, independent of spin.

where

$$\mathcal{O}^{\mu\nu}{}_{\rho\lambda}(\tau) = \delta^\mu_\rho \delta^\nu_\lambda \;+\; \frac{2e}{m}\int_{-\infty}^{\tau} d\tau'\, \delta_\rho{}^{[\mu} F^{\nu]}{}_\lambda(\tau') + \frac{e^2}{2m^2}\int_{-\infty}^{\tau} d\tau' \int_{-\infty}^{\tau} d\tau''\, F^{\mu\nu}(\tau') F_{\rho\lambda}(\tau'') \\ +\; \frac{e^2}{(n\cdot p)^2} n^{[\mu}\delta^{\nu]}{}_\rho n_\lambda A^2(\tau)\,. \tag{8}$$

Note that the above expression for the spin tensor involves the vector potential in the Lorentz gauge. However, $\mathcal{S}$ is invariant under the restricted gauge transformation $A_\mu(y) \to A_\mu(y) + \partial_\mu \chi(y)$, as one may easily verify.

This completely defines the dynamics of the spin tensor in this specific background field. For additional details and for the solution in the gravitational wave background see Ref. 1. Let us turn next to the issue of constraints. Any set of dynamical equations for the spin tensor is incomplete without conditions which restrict the number of components of the spin tensor.[3,4] In this case, one can show that $\mathcal{S}$ satisfies the following constraints without the use of equations of motion, namely

$$v_\mu(\tau)\mathcal{S}^{\mu\nu}(\tau) \equiv \frac{1}{m}(p_\mu - eA_\mu + n_\mu I_p(\tau))\mathcal{S}^{\mu\nu}(\tau) = 0 \tag{9}$$

assuming $v_\mu(\tau \to -\infty)\mathcal{S}^{\mu\nu}(\tau \to -\infty) = p_\mu/m\ \mathcal{S}^{\mu\nu}(\tau \to -\infty) = 0$, and

$$\frac{d}{d\tau}\left(\mathcal{S}^{\mu\nu}(\tau)\mathcal{S}_{\mu\nu}(\tau)\right) = 0\,. \tag{10}$$

Acknowledgements

I am indebted to David Boulware, Lowell Brown, and Laurence Yaffe for many useful discussions. This work was supported in part by DOE grant DE-FG09-91ER40614 .

References

1. S. Vokos, *"Motion of Spin–$\frac{1}{2}$ Particles in Gravitational Fields: Spin-Curvature Interaction"*, preprint UW/PT-94-12, in preparation.
2. L. S. Brown and T. W. B. Kibble, *Phys. Rev.* **133** (1964) A705.
3. A. Papapetrou, *Proc. Roy. Soc. Lond. A* **209** (1951) 248.
4. W. G. Dixon, *Proc. Roy. Soc. Lond. A* **314** (1970) 499.